1. CLIFFORD ALGEBRAS

Throughout this text, let K denote the field $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. The *tensor algebra on* K^n is defined to be the algebra

$$T(K^n) = \oplus_{r=0}^{\infty} T^r(K^n) = K \oplus K^n \oplus K^n \otimes K^n \oplus \cdots$$

whose multiplication is given by linearly extending the canonical map

$$T^\ell(K^n) \times T^m(K^n) \to T^{\ell+m}(K^n).$$

Note $T(K^n)$ is associative with multiplicative identity $1 \in K = T^0(K^n) \subset T(K^n)$.

Definition 1.2. The *Clifford Algebra on* K^n is defined to be the quotient of algebras

$$\mathcal{C}l(K^n) = T(K^n)/I(K^n)$$

where $I(K^n)$ is the two-sided ideal generated by elements of form

$$v \otimes v + < v, v > 1 \in T(K^n) \quad \text{with} \quad v \in K^n.$$

Here $<\,,>$ denotes the usual inner product on K^n, also known as the dot product.

Note $\mathcal{C}l(K^n)$ has the multiplicative identity, $[1] \in [K] \subset \mathcal{C}l(K^n)$. We will denote $[1]$ with 1.

Lemma 1.1. *There is a natural embedding of* $K^n = T^1(K^n)$ *into* $\mathcal{C}l(K^n)$.

Proof. It is sufficient to show $K^n \cap I(K^n) = \{0\}$. First an element in $T(K^n)$ is said to be of *pure degree* s if it is contained in $T^s(K^n) \subset T(K^n)$. Let $\varphi \in K^n \cap I(K^n)$. Since $\varphi \in I(K^n)$ we may write it as a finite sum

$$\varphi = \sum_i a_i \otimes (v_i \otimes v_i + < v_i, v_i >) \otimes b_i$$

where the a_i and b_i are each of pure degree. Since $\varphi \in K^n = T^1(K^n)$, we have

$$\sum_{i'} a_{i'} \otimes (v_{i'} \otimes v_{i'}) \otimes b_{i'} = 0$$

where the above sum is taken over indices where each $\deg a_{i'} + \deg b_{i'}$ is maximal. By contraction with $<\,,>$, we also have

$$\sum_{i'} a_{i'} < v_{i'}, v_{i'} > b_{i'} = 0.$$

Hence

$$\sum_{i'} a_{i'} \otimes (v_{i'} \otimes v_{i'} + < v_{i'}, v_{i'} >) \otimes b_{i'} = 0.$$

and proceeding inductively, we conclude $\varphi = 0$. □

Lemma 1.2. *Let $\{e_1, \cdots, e_n\}$ denote an orthonormal basis for K^n. Then $\mathcal{C}l(K^n)$ is generated as an algebra by the e_i subject to the relations:*

$$e_i^2 = -1 \quad \textit{and} \quad e_i e_j = -e_j e_i$$

for each i and $j \neq i$.

It follows $\mathcal{C}l(K^n)$ is a 2^n-dimensional vector space with basis

$$\{1\} \cup \{e_{i_1} e_{i_2} \cdots e_{i_r} \mid 1 \leq i_1 < i_2 < \cdots i_r \leq n\}.$$

□

For the reader's convenience, we recall the definition of the complexification of a real algebra.

Definition 1.3. If A is an algebra over $\mathbb{R}$ then its complexification is the algebra $A \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the following complex scalar multiplication map. For each $v \otimes z \in A \otimes_{\mathbb{R}} \mathbb{C}$ and $\lambda \in \mathbb{C}$, we define $\lambda(v \otimes z) = v \otimes (\lambda z)$.

Lemma 1.3. *The complexification of $\mathcal{C}l(\mathbb{R}^n)$, $\mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C}$, is isomorphic as an algebra to $\mathcal{C}l(\mathbb{C}^n)$.*

Proof. Define an algebra isomorphism $\mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C} \to \mathcal{C}l(\mathbb{C}^n)$ by

$$v \otimes \lambda \mapsto \lambda v \in \mathbb{C}^n \subset Cl(\mathbb{C}^n)$$

for each $v \in \mathbb{R}^n \subset \mathcal{C}l(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$.

□

Lemma 1.4. *We can construct an algebra isomorphism:*

$$\mu : \mathcal{C}l(\mathbb{C}^4) \to Mat(\mathbb{C}, 4).$$

Proof. Let $\{e_1, \cdots, e_4\}$ denote the usual orthonormal basis for $\mathbb{C}^4 \subset \mathcal{C}l(\mathbb{C}^4)$. We can define an algebra isomorphism $\mathcal{C}l(\mathbb{C}^4) \to Mat(2, \mathbb{C}) \otimes Mat(2, \mathbb{C})$ as follows.

$$e_1 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \otimes \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad e_2 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$e_3 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad e_4 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

By composing this map with the Kronecker map, we obtain the desired algebra isomorphism, μ. □

Using the map μ from the previous Lemma, we have an action of $\mathcal{C}l(\mathbb{C}^4)$ on $\mathbb{C}^4$ which we will refer to as *Clifford multiplication* . We will denote this action by "$\cdot$".

Now we will discuss the splittings of Clifford algebras. First note by Lemma 1.2, the map $\alpha : K^n \to K^n$ given by $\alpha(v) = -v$ extends to an algebra automorphism of $\mathcal{C}l(K^n)$. α induces the splitting

$$\mathcal{C}l(K^n) = \mathcal{C}l_0(K^n) \oplus \mathcal{C}l_1(K^n)$$

where $\mathcal{C}l_i(K^n) = \{\varphi \in \mathcal{C}l(K^n) \mid \alpha(\varphi) = (-1)^i\varphi\}$. To verify this is indeed a splitting note each $Cl_i(K^n)$ is a linear subspace, $span\{Cl_0(K^n), Cl_1(K^n)\} = Cl(K^n)$ as each element of the vector space basis $\{1\} \cup \{e_{i_1}e_{i_2}\cdots e_{i_r} \mid 1 \leq i_1 < i_2 < \cdots i_r \leq n\}$ is contained in one of the $Cl_i(K^n)$, and $Cl_0(K^n) \cap Cl_1(K^n) = 0$ since if $\varphi \in Cl_0(K^n) \cap Cl_1(K^n)$ then $\varphi = -\varphi$ implies $\varphi = 0$ as K has no nontrival torsion.

The *volume element* of $Cl(\mathbb{R}^n)$ (oriented by the usual orientation of $\mathbb{R}^n$) is defined to be $\omega = e_1 \cdots e_n$ where the e_i are an orthonormal basis of $\mathbb{R}^n$ with the usual orientation. To see this is well-defined, suppose $e'_1, \cdots, e'_n$ is another oriented orthonormal basis. Then each $e'_i = \sum_j g_{ij}e_j$ for some $g = (g_{ij}) \in SO(n)$. Hence from Lemma 1.2

$$e'_1 \cdots e'_n = \det(g)e_1 \cdots e_n = e_1 \cdots e_n.$$

In the case $\omega^2 = 1$, we have the splitting

$$\mathcal{C}l(\mathbb{R}^n) = \mathcal{C}l^+(\mathbb{R}^n) \oplus \mathcal{C}l^-(\mathbb{R}^n)$$

where $\mathcal{C}l^\pm(\mathbb{R}^n) = \pi^\pm\mathcal{C}l(\mathbb{R}^n)$ with $\pi^\pm = \frac{1}{2}(1 \pm \omega)$ (note we use this notation since left multiplication by $\pi^\pm$ is a projection onto $\mathcal{C}l^\pm(\mathbb{R}^n)$) . To see this is a splitting, observe each $Cl^\pm(\mathbb{R}^n)$ is a linear subspace, $span\{Cl^+(\mathbb{R}^n), Cl^-(\mathbb{R}^n)\} = Cl(K^n)$ since we can write each $\varphi \in Cl(\mathbb{R}^n)$ as $\varphi = \pi^+\varphi + \pi^-\varphi$, and $Cl^+(\mathbb{R}^n) \cap Cl^-(\mathbb{R}^n) = 0$ since if $\pi^+\varphi_1 = \pi^-\varphi_2$, using the facts $\pi^\pm\pi^\pm = \pi^\pm$ and $\pi^\pm\pi^\mp = 0$, we obtain $0 = \pi^-\varphi_2$. Note for each $e \in \mathbb{R}^n$, we have $\pi^\pm e = e\frac{1}{2}(1 \mp \omega)$.

Lemma 1.5.

$$\mathcal{C}l(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}$$

Proof. Define an algebra isomorphism $\mathbb{H} \oplus \mathbb{H} \to \mathcal{C}l(\mathbb{R}^3)$ by

$$i \oplus 0 \mapsto \tfrac{1}{2}(e_1e_2 - e_3) \quad j \oplus 0 \mapsto \tfrac{1}{2}(e_2e_3 - e_1)$$

$$0 \oplus i \mapsto \tfrac{1}{2}(e_1e_2 + e_3) \quad 0 \oplus j \mapsto \tfrac{1}{2}(e_2e_3 + e_1)$$

where the e_i are the usual oriented orthonormal basis for $\mathbb{R}^3 \subset Cl(\mathbb{R}^3)$. □

Note this map sends $\mathbb{H} \oplus 0$ to $Cl^+(\mathbb{R}^3)$ and $0 \oplus \mathbb{H}$ to $Cl^-(\mathbb{R}^3)$.

Lemma 1.6.

$$\mathcal{C}l(K^{n-1}) \cong \mathcal{C}l_0(K^n)$$

Proof. We can define an algebra isomorphism $\mathcal{C}l(K^{n-1}) \to \mathcal{C}l_0(K^n)$ by

$$e_i \mapsto e_i e_n$$

where the e_i on the left are the usual orthonormal basis for $K^{n-1} \subset Cl(K^{n-1})$ and the e_i on the right are the usual orthonormal basis for $K^n \subset Cl(K^n)$.

□

Note the isomorphism from the previous Lemma sends the ω of $\mathcal{C}l(\mathbb{R}^3)$ to the ω of $\mathcal{C}l(\mathbb{R}^4)$ so it preserves the corresponding splittings, i.e. $\mathcal{C}l^{\pm}(\mathbb{R}^3) \mapsto \mathcal{C}l_0^{\pm}(\mathbb{R}^4)$.

Similarly $Cl(\mathbb{C}^n)$ has a *complex volume element* (oriented by the usual orientation of $\mathbb{R}^n$), $\omega_{\mathbb{C}} = i^{\left[\frac{n(n-1)}{2}\right]} e_1 \cdots e_n$ where the e_i are an oriented orthonormal basis for $\mathbb{R}^n = (\mathbb{R} \oplus 0) \oplus \cdots \oplus (\mathbb{R} \oplus 0) \subset \mathbb{C}^n \subset Cl(\mathbb{C}^n)$. Note $\omega_{\mathbb{C}}^2 = 1$ so it induces a splitting

$$\mathcal{C}l(\mathbb{C}^n) = \mathcal{C}l^+(\mathbb{C}^n) \oplus \mathcal{C}l^-(\mathbb{C}^n)$$

where $\mathcal{C}l^{\pm}(\mathbb{C}^n) = \pi_{\mathbb{C}}^{\pm} \mathcal{C}l(\mathbb{C}^n)$ with $\pi_{\mathbb{C}}^{\pm} = \frac{1}{2}(1 \pm \omega_{\mathbb{C}})$.

Finally note Clifford multiplication induces a splitting

$$\mathbb{C}^4 = (\mathbb{C}^4)^+ \oplus (\mathbb{C}^4)^-$$

where $(\mathbb{C}^4)^{\pm} = \pi_{\mathbb{C}}^{\pm} \cdot \mathbb{C}^4$.

Lemma 1.7. $\dim(\mathbb{C}^4)^{\pm} = 2$ *and Clifford multiplication induces vector space isomorphisms:*

$$\begin{aligned} \mathbb{C}^4 \to &\quad Hom_{\mathbb{C}}((\mathbb{C}^4)^+, (\mathbb{C}^4)^-) \\ \mathbb{C}^4 \to &\quad Hom_{\mathbb{C}}((\mathbb{C}^4)^-, (\mathbb{C}^4)^+). \end{aligned}$$

Proof. First define a linear maps $\phi_{\pm} : \mathbb{C}^4 \to Hom_{\mathbb{C}}((\mathbb{C}^4)^{\pm}, (\mathbb{C}^4)^{\mp})$ by $e \mapsto (v \mapsto e \cdot v)$ where $e \in \mathbb{C}^4 \subset \mathcal{C}l(\mathbb{C}^4)$ and $v \in (\mathbb{C}^4)^{\pm} \subset \mathbb{C}^4$.

To see this is well-defined, choose $v \in (\mathbb{C}^4)^{\pm}$ and $e \in \mathbb{C}^4 \subset \mathcal{C}l(\mathbb{C}^4)$. Then

$$e \cdot v = e\pi_{\mathbb{C}}^{\pm} \cdot v = \pi_{\mathbb{C}}^{\mp} e \cdot v \in (\mathbb{C}^4)^-.$$

For any $e \in \mathbb{C}^4 \subset \mathcal{C}l(\mathbb{C}^4)$ with $< e, e > \neq 0$ and $v \in (\mathbb{C}^4)^{\pm}$,

$$\phi_{\mp}(e) \circ \phi_{\pm}(e)(v) = - < e, e > v$$

is an automorphism of $(\mathbb{C}^4)^{\pm}$ hence $\dim(\mathbb{C}^4)^+ = \dim(\mathbb{C}^4)^-$. Then since $\dim(\mathbb{C}^4)^+ + \dim(\mathbb{C}^4)^- = 4$, $\dim(\mathbb{C}^4)^{\pm} = 2$.

Now we will show $\phi_\pm$ are injective. First suppose $\phi_+(e) = 0$ for some $e \in \mathbb{C}^4 \subset \mathcal{C}l(\mathbb{C}^4)$ then $e \cdot v = 0$ for each $v \in (\mathbb{C}^4)^+$. It follows $e\pi_{\mathbb{C}}^+ \cdot v' = 0$ for each $v' \in \mathbb{C}^4$. Since Clifford multiplication is faithful, we have $e\pi_{\mathbb{C}}^+ = 0$. Hence $\pi_{\mathbb{C}}^- e = 0$. We may write $e = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4$ where the e_i are the usual orthonormal basis for $\mathbb{C}^4$ and the c_i are constants in $\mathbb{C}$. Then the previous equation may be written

$$\pi_{\mathbb{C}}^-(c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4) = 0.$$

If we distribute, this turns into

$$\frac{1}{2}(c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_1e_2e_3e_4 - c_2e_1e_3e_4 + c_3e_1e_2e_4 - c_4e_1e_2e_3) = 0.$$

This is a linear combination of distinct basis vectors so each $c_i = 0$ and hence $e = 0$. Proving ϕ_- is injective can be done similarly.

Therefore since $\dim Hom_{\mathbb{C}}((\mathbb{C}^4)^\pm, (\mathbb{C}^4)^\mp) = 4$, $\phi_\pm$ are isomorphisms.

□

Using this Lemma, we can define maps $C : \mathbb{C}^4 \otimes (\mathbb{C}^4)^\pm \to (\mathbb{C}^4)^\mp$ which we will also refer to as *Clifford multiplication*.

Lemma 1.8. *Clifford multiplication induces algebra isomorphisms*

$$\mathcal{C}l_0^\pm(\mathbb{C}^4) \to End(\mathbb{C}^4)^\pm.$$

Proof. Define algebra isomorphisms $\phi_\pm : \mathcal{C}l_0^\pm(\mathbb{C}^4) \to End(\mathbb{C}^4)^\pm$ by $\varphi \mapsto (v \mapsto \varphi \cdot v)$ where $\varphi \in Cl_0^\pm(\mathbb{C}^4)$ and $v \in (\mathbb{C}^4)^\pm \subset \mathbb{C}^4$. Since $\varphi = \pi_{\mathbb{C}}^\pm \varphi$ for all $\varphi \in Cl^\pm(\mathbb{C}^4)$, this is well-defined.

To see $\phi^\pm$ are injective suppose $\varphi \cdot v = 0$ for all $v \in (\mathbb{C}^4)^\pm$. Then $\varphi \cdot \pi_{\mathbb{C}}^\pm \cdot v' = 0$ for each $v' \in \mathbb{C}^4$. Because $\omega_{\mathbb{C}}$ commutes with each element in $\mathcal{C}l_0(\mathbb{C}^4)$, we have $\varphi \cdot v' = 0$ and since Clifford multiplication is faithful, $\varphi = 0$.

Then since $\dim Cl_0^\pm(\mathbb{C}^4) = \dim End(\mathbb{C}^4)^\pm = 4$, $\phi_\pm$ are vector space isomorphisms. Additionally they are algebra isomorphisms since for each $\varphi \in \mathcal{C}l_0^\pm(\mathbb{C}^4)$,

$$\phi_\pm(\varphi_1\varphi_2)(v) = \varphi_1\varphi_2 \cdot v = \varphi_1 \cdot (\varphi_2 \cdot v) = \phi_\pm(\varphi_1) \circ \phi_\pm(\varphi_2)(v).$$

□

2. Exterior algebras

First we define the vector space of *alternating tensors* of degree r on K^n to be
$\wedge^r(K^n) = \{\varphi \in T^r(K^n) \mid \varphi(v_1, \cdots, v_i, \cdots, v_j, \cdots v_r) = -\varphi(v_1, \cdots, v_j, \cdots, v_i, \cdots v_r)\}$.

We have the *alternating projection* map $Alt : T^r(K^n) \to \wedge^r(K^n)$ given by

$$\varphi(v_1, \cdots, v_r) \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} sgn(\sigma)\varphi(v_{\sigma(1)}, \cdots, v_{\sigma(r)})$$

where S_r denotes the set of permutations of r elements.

Definition 2.1. The *exterior algebra* on K^n is defined to be the algebra

$$\wedge(K^n) = \oplus_{r=0}^{\infty} \wedge^r (K^n) = K \oplus K^n \oplus \wedge^2(K^n) \oplus \wedge^3(K^n) \oplus \cdots$$

whose multiplication $\wedge$ is given by linearly extending the map $\wedge^\ell(K^n) \otimes \wedge^m(K^n) \to \wedge^{\ell+m}(K^n)$ defined by

$$(\varphi, \phi) \mapsto \frac{(\ell+m)!}{\ell! m!} Alt(\varphi \otimes \phi).$$

Note $\wedge(K^n)$ is an associative algebra with multiplicative identity $1 \in K = \wedge^0(K^n) \subset \wedge(K^n)$.

Lemma 2.1. *Let $\{e_1, \cdots, e_n\}$ denote a basis for K^n. Then $\wedge(K^n)$ is generated as an algebra by the e_i and 1. It follows $\wedge(K^n)$ is a 2^n-dimensional vector space with basis*

$$\{1\} \cup \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} \mid 1 \leq i_1 < i_2 < \cdots i_r \leq n\}.$$

□

Corollary 2.1. $\wedge(K^n)$ and $\mathcal{C}l(K^n)$ are naturally isomorphic as vector spaces.

We define the *Hodge star operator* (induced by the usual orientation and the usual inner product $<,>$) to be the map

$$* : \wedge^2(\mathbb{R}^n) \to \wedge^2(\mathbb{R}^n)$$

given by $*(e_i \wedge e_j) = e_k \wedge e_\ell$ where the e_i are an oriented orthonormal basis and (i, j, k, ℓ) is an even permutation of $\{1, 2, 3, 4\}$. Hence

$$\begin{array}{ll} e_1 \wedge e_2 \mapsto e_3 \wedge e_4 & e_1 \wedge e_3 \mapsto -e_2 \wedge e_4 \\ e_1 \wedge e_4 \mapsto e_2 \wedge e_3 & e_2 \wedge e_3 \mapsto e_1 \wedge e_4 \\ e_2 \wedge e_4 \mapsto -e_1 \wedge e_3 & e_3 \wedge e_4 \mapsto e_1 \wedge e_2 \end{array}$$

Lemma 2.2. *The Hodge star operator is well-defined.* □

$* : \wedge^2(\mathbb{R}^n) \to \wedge^2(\mathbb{R}^n)$ induces a splitting

$$\wedge^2(\mathbb{R}^4) = \wedge^-(\mathbb{R}^4) \oplus \wedge^+(\mathbb{R}^4)$$

where $\wedge^\pm(\mathbb{R}^4) = \{\varphi \in \wedge^2(\mathbb{R}^4) \mid *\varphi = \pm\varphi\}$.

Lemma 2.3. *The natural isomorphism from Corollary 2.1 induces a vector space isomorphism between the subspaces*

$$(Cl_0(\mathbb{R}^4) \otimes \mathbb{C})^+ \text{ and } \pi^+_{\mathbb{C}}\mathbb{C} \oplus (\wedge^+(\mathbb{R}^4) \otimes \mathbb{C}).$$

Proof. Let $\{e_1, \cdots, e_4\}$ denote an oriented orthonormal basis for $\mathbb{R}^4 = \mathbb{R}^4 \otimes 1$. Notice $(Cl_0(\mathbb{R}^4) \otimes \mathbb{C})^+$ has the basis

$$\left\{\pi^+_{\mathbb{C}}, e_1e_2 + e_3e_4, e_1e_3 - e_2e_4, e_1e_4 + e_2e_3\right\}$$

so we see $(Cl_0(\mathbb{R}^4) \otimes \mathbb{C})^+ \mapsto \pi^+_{\mathbb{C}}\mathbb{C} \oplus \wedge^+(\mathbb{R}^4) \otimes \mathbb{C}$. □

3. Spin(n) and Spinc(n)

Now we will define the Lie groups $Spin(n)$ and $Spin^c(n)$. Let $\mathcal{C}l^\times(\mathbb{R}^n)$ denote the multiplicative group of units in $\mathcal{C}l(\mathbb{R}^n)$. We define $Pin(n)$ to be the subgroup of $\mathcal{C}l^\times(\mathbb{R}^n)$ generated by elements $v \in \mathbb{R}^n \subset Cl(\mathbb{R}^n)$ with $< v, v >= 1$. We define $Spin(n)$ to be the intersection of $Pin(n)$ and $\mathcal{C}l_0(\mathbb{R}^n)$.

Lemma 3.1. *We have an isomorphism $SU(2) \times SU(2) \to Spin(4)$ where the splitting on the left corresponds with the splitting $Spin(4) = Spin(4)^+ \times Spin(4)^-$.*

Proof. Recall we have algebra isomorphisms

$$\mathbb{H} \oplus \mathbb{H} \to \mathcal{C}l(\mathbb{R}^3) \to \mathcal{C}l_0(\mathbb{R}^4)$$

where the first map is the isomorphism from Lemma 1.5 and the second map is the isomorphism from Lemma 1.6.

If we identify $\mathbb{H}$ with $\mathbb{R}^4$, the group of unit quaternions is identified with $S^3 \subset \mathbb{R}^4$ and is isomorphic to $SU(2)$. Then by restricting the composition of the above maps to $S^3 \times S^3 \subset \mathbb{H} \oplus \mathbb{H}$, we obtain an isomorphism

$$SU(2) \times SU(2) \to Spin(4).$$

This composition preserves the desired splittings as each of the isomorphisms above preserves its own corresponding splittings.

□

We define $Spin^c(n)$ to be the multiplicative group of units $[Spin(n) \times S^1] \subset \mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C}$. Observe by using the algebra isomorphism from Lemma 1.3 , we can consider $Spin^c(n)$ to be a multiplicative group of units contained in $\mathcal{C}l(\mathbb{C}^4)$.

Lemma 3.2. $Spin^c(n) \cong Spin(n) \times S^1/\{\pm(1, 1)\}$.

Proof. First we have the natural surjective group homomorphism

$$Spin(n) \times S^1 \hookrightarrow [Spin(n) \times S^1].$$

Elements of the kernel of this map are of form $(c1, c^{-1})$ where $c \in S^1 \cap \mathbb{R} = \{-1, 1\}$ AND $c1 \in Spin(n)$. To see $-1 \in Spin(n) = Pin(n) \cap Cl_0(\mathbb{R}^n)$, first $-1 \in Pin(n)$ since $e_1 e_1 = -1$ and $-1 \in \mathcal{C}l_0(\mathbb{R}^n)$ since

$$\alpha(-1) = \alpha(e_1 e_1) = \alpha(e_1)\alpha(e_1) = (-e_1)(-e_1) = e_1 e_1 = -1.$$

Thus the kernel of our map is $\{\pm(1, 1)\}$ so by the first isomorphism theorem,

$$Spin^c(n) \cong Spin(n) \times S^1/\{\pm(1, 1)\}.$$

□

From this Lemma and Lemma 3.1, we obtain an isomorphism

$$Spin^c(4) \cong SU(2) \times SU(2) \times S^1/\{\pm(I, I, 1)\}.$$

Note that under this isomorphism, $Spin^c(4)^+ \oplus \pi_{\mathbb{C}}^- \subset \mathcal{C}l(\mathbb{C}^4)$ is identified with the subgroup $[SU(2) \times I \times S^1] \subset SU(2) \times SU(2) \times S^1/\{\pm(I, I, 1)\}$ and $Spin^c(4)^- \oplus \pi_{\mathbb{C}}^+$ is identified with the subgroup $[I \times SU(2) \times S^1]$.

Lemma 3.3. *We have a group isomorphism*

$$\{(A, B) \in U(2) \times U(2) \ \mid \det(A) = \det(B)\} \to Spin^c(4) \subset \mathcal{C}l(\mathbb{C}^4)$$

where the splitting of $\{(A, B) \in U(2) \times U(2) \ \mid \det(A) = \det(B)\}$ *corresponds with the splitting* $Spin^c(4) = Spin^c(4)^+ \times Spin^c(4)^-$.

Proof. First there is an isomorphism

$$\{(A, B) \in U(2) \times U(2) \ \mid \det(A) = \det(B)\} \to SU(2) \times SU(2) \times S^1/\{\pm(I, I, 1)\}$$

defined by

$$(A, B) \mapsto [(A \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, B \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \lambda)]$$

where $\lambda^2 = \det A$. Note since

$$[(A, B, \lambda)] = [(-A, -B, -\lambda)] \quad \text{in} \quad SU(2) \times SU(2) \times S^1/\{\pm(I, I, 1)\},$$

our map is the same for each of the two choices of λ and hence is well-defined.

The rest follows from the comments after Lemma 3.2.

□

The *adjoint representation* of $Spin(n)$ is the map $Ad : Spin(n) \to Aut(\mathcal{C}l(\mathbb{R}^n))$ defined by $\varphi \mapsto (y \mapsto \varphi y \varphi^{-1})$. Recall

$$Pin(n) = \{v_1 \cdots v_r \in \mathcal{C}l(\mathbb{R}^n) \mid v_i \in \mathbb{R}^n \text{ with } < v_i, v_i >= 1\}$$

so we see $Ad_\varphi(v) \in \mathbb{R}^n$ for each $\varphi \in Spin(n) \subset Pin(n)$ and $v \in \mathbb{R}^n$. Hence we can restrict the range to obtain a homomorphism $Ad : Spin(n) \to GL(n)$. In fact:

Lemma 3.4. *Ad induces a group homomorphism,*

$$\xi : Spin(n) \to SO(n)$$

which is a double covering map. For $n > 2$, this is the universal double cover (up to isomorphism). ☐

For $Spin^c(n)$, we can define a double-covering map of $SO(n) \times U(1)$ as follows. Let $\xi^c : Spin^c(n) \to SO(n) \times U(1)$ be the homomorphism $[(\varphi, \lambda)] \mapsto (\xi(\varphi), \lambda^2)$. Also observe the map $\xi : Spin(n) \to SO(n)$ induces the homomorphism $\xi : Spin^c(n) \to SO(n)$ given by $[(\varphi, \lambda)] \mapsto \xi(\varphi)$. The kernel of this map is $Z(Spin^c(n)) \cong S^1$.

Lemma 3.5.

$$Spin^c(n) \cong Spin^c(n) \times_{S^1} S^1 = Spin^c(n) \times S^1 / \{(\lambda 1, \lambda^{-1}) \mid \lambda \in S^1\}$$

Proof. Define an isomorphism by $\varphi \mapsto [\varphi, 1]$. To see this is onto observe $[\varphi, \lambda] = [\lambda\varphi, 1]$ for each $\varphi \in Spin^c(n)$ and $\lambda \in S^1$. To see injectivity suppose $[\varphi_1, 1] = [\varphi_2, 1]$ for some $\varphi_i \in Spin^c(n)$. Then $(\varphi_1, 1) = (\lambda\varphi_2, \lambda^{-1})$ for some $\lambda \in S^1$. Hence $\lambda = 1$ and $\varphi_1 = \varphi_2$. ☐

4. SPINC STRUCTURES

Given an orientable manifold X, recall a choice of orientation and Riemannian metric reduces the structure group of TX to $SO(n) \subset GL(n)$ hence we obtain a frame bundle $P_{SO(n)}$.

Definition 4.1. A *$Spin^c$-structure* for an oriented Riemannian n-manifold X is a principal $Spin^c(n)$-bundle $P_{Spin^c(n)} \to X$ together with a bundle map $P_{Spin^c(n)} \to P_{SO(n)}$ that is $\xi\colon Spin^c(n) \to SO(n)$ fibrewise.

The *determinant line bundle* of a $Spin^c$-structure $P_{Spin^c(n)} \to P_{SO(n)}$ is defined to be the complex line bundle $L = P_{Spin^c(n)} \times_{\det} \mathbb{C}$ where $\det : Spin^c(n) \to U(1)$ is given by $[(\varphi, \lambda)] \mapsto \lambda^2$.

Now we will restrict our attention to the four-dimensional case. Using the Clifford multiplication map $\mu : \mathcal{C}l(\mathbb{C}^4) \to Mat(\mathbb{C}, 4)$ from Lemma 1.4, the *complex spinor bundle* associated to μ is defined to be the complex vector bundle $W = P_{Spin^c(n)} \times_\mu \mathbb{C}^4$.

We may split W as $W = W^+ \oplus W^-$ where

$$W^\pm = P_{Spin^c(4)} \times_{\mu^\pm} (\mathbb{C}^4)^\pm$$

where $\mu^\pm(\bullet) = \mu(\pi^\pm_{\mathbb{C}} \bullet)$. W^+ is called the *positive complex spinor bundle* and W^- is called the *negative complex spinor bundle*. From Lemma 3.3, both $W^\pm$ have structure group $U(2)$.

Now we will show $H^2(X;\mathbb{Z})$ has an action on $Spin^c(X)$ (the set of isomorphism classes of $Spin^c$-structures on X). For $E \in H^2(X;\mathbb{Z})$, let $P_{U(1)}$ denote the corresponding principal $U(1)$-bundle. We can define a new $Spin^c$-structure $\xi \otimes E$ as follows. Consider

$$P_{Spin^c(4)} \times_{U(1)} P_{U(1)} = P_{Spin^c(4)} \times P_{U(1)}/\sim$$

where $(\varphi, y) \sim (\varphi \cdot \lambda, y \cdot \lambda^{-1})$ for each $\lambda \in U(1)$. On the left, $U(1)$ is identified with $Z(Spin^c(n))$ in our usual way. From Lemma 3.5, this is a principal $Spin^c(n)$ bundle. We can define our bundle map $P_{Spin^c(4)} \times_{U(1)} P_{U(1)} \to P_{SO(n)}$ by $[\varphi, y] \mapsto \xi(\varphi)$.

Finally observe the induced map $\det : Spin^c(n) \times_{S^1} S^1 \to S^1$ is given by $[\varphi \otimes z, \lambda] \mapsto z^2\lambda^2$. We can write this as $\det = \det_1 \det_2 \det_2$ where $\det_i : Spin^c(n) \times_{S^1} S^1 \to S^1$ are given by $det_1([\varphi \otimes z, \lambda]) = z^2$ and $det_2([\varphi \otimes z, \lambda]) = \lambda$. Hence

$$\begin{aligned}
(P_{Spin^c(4)} \times_{U(1)} P_{U(1)}) \times_{\det} \mathbb{C} = &\ (P_{Spin^c(4)} \times_{U(1)} P_{U(1)}) \times_{\det} \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \\
= &\ ((P_{Spin^c(4)} \times_{U(1)} P_{U(1)}) \times_{\det_1} \mathbb{C}) \otimes \\
&\ ((P_{Spin^c(4)} \times_{U(1)} P_{U(1)}) \times_{\det_2} \mathbb{C}) \otimes \\
&\ ((P_{Spin^c(4)} \times_{U(1)} P_{U(1)}) \times_{\det_2} \mathbb{C}) \\
= &\ L \otimes E \otimes E
\end{aligned}$$

So we see our action has the following effect on determinant line bundles: $L \mapsto L + 2E$.

Lemma 4.1. *The above action is free and transitive.* □

5. Clifford bundles

Definition 5.1. Given a oriented Riemannian n-manifold X with frame bundle $P_{SO(n)}$, we define the *Clifford bundle* of X as $\mathcal{C}l(X) = P_{SO(n)} \times_{SO(n)} \mathcal{C}l(\mathbb{R}^n)$. We also have the complexified Clifford bundle $Cl(X) \otimes \mathbb{C} = P_{SO(n)} \times_{SO(n)} (\mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C})$.

Let X be a oriented Riemannian n-manifold with frame bundle $P_{SO(n)}$ and $Spin^c$-structure $\xi : P_{Spin^c(n)} \to P_{SO(n)}$.

Lemma 5.1. *The map $\xi : P_{Spin^c(n)} \to P_{SO(n)}$ induces a bundle isomorphism:*

$$P_{Spin^c(n)} \times_{Ad} \mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C} \to \mathcal{C}l(X) \otimes \mathbb{C}$$

where $Ad : Spin^c(n) \to Aut(\mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C})$ is given by

$$\varphi \otimes \lambda \mapsto (y \otimes v \mapsto \varphi y \varphi^{-1} \otimes \lambda v \lambda^{-1} = \varphi y \varphi^{-1} \otimes v).$$

Proof. Define a map

$$P_{Spin^c(n)} \times \mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C} \to P_{SO(n)} \times \mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C}$$

by $(y, v) \mapsto (\xi(y), v)$. For $\varphi \otimes \lambda \in Spin^c(n)$ and $(y, v \otimes z) \in P_{Spin^c(n)} \times \mathcal{C}l(\mathbb{R}^n) \otimes \mathbb{C}$, we have

$$(y \cdot \varphi^{-1} \otimes \lambda^{-1}, \varphi v \varphi^{-1} \otimes z) \mapsto (\xi(y) \cdot \xi(\varphi)^{-1}, \xi(\varphi) v \otimes z)$$

so our map induces a bundle map

$$\xi' : P_{Spin^c(n)} \times_{Ad} \mathcal{C}l(\mathbb{R}^4) \otimes \mathbb{C} \to \mathcal{C}l(X) \otimes \mathbb{C}.$$

Surjectivity follows from the fact that ξ is onto. To see ξ' is injective suppose

$$\xi'([y_1, v_1 \otimes z_1]) = \xi'([y_2, v_2 \otimes z_2]).$$

Then $[(\xi(y_1), v_1 \otimes z_1)] = [(\xi(y_2), v_2 \otimes z_2)]$ and hence

$$(\xi(y_1) \cdot \xi(\varphi)^{-1}, \xi(\varphi)v_1 \otimes z_1) = (\xi(y_2), v_2 \otimes z_2)$$

for some $\varphi \otimes \lambda \in Spin^c(n)$. Since $Spin^c(n)$ acts transitively on the fibres of $P_{Spin^c(n)}$, we have $y_1 \cdot (\varphi' \otimes \lambda')^{-1} = y_2$ for some $\varphi' \otimes \lambda' \in Spin^c(4)$. Observe

$$\xi(y_2) = \xi(y_1 \cdot (\varphi' \otimes \lambda')^{-1}) = \xi(y_1) \cdot \xi(\varphi')^{-1}$$

so since $SO(n)$ acts freely on the fibres of $P_{SO(n)}$, it follows $\xi(\varphi') = \xi(\varphi)$ and hence

$$(y_1 \cdot (\varphi' \otimes \lambda')^{-1}, \xi(\varphi')v_1 \otimes z_1) = (y_2, v_2 \otimes z_2)$$

and therefore ξ' is a bundle isomorphism.

□

Now additionally suppose X is 4-dimensional with complex spinor bundle $W = W^+ \oplus W^-$. We'll show $Cl(X) \otimes \mathbb{C}$ has an action called Clifford multiplication on W. Define a map

$$C : P_{Spin^c(4)} \times (\mathcal{C}l(\mathbb{C}^4) \otimes \mathbb{C}^4) \to P_{Spin^c(4)} \times \mathbb{C}^4$$

by $(q, \varphi \otimes v) \mapsto (q, \varphi \cdot v)$ where $\cdot$ denotes Clifford multiplication. For $g \in Spin^c(4)$, we have

$$C(qg^{-1}, g\varphi g^{-1} \otimes g \cdot v) = (qg^{-1}, g\varphi g^{-1} \cdot g \cdot v) = (qg^{-1}, g \cdot (\varphi \cdot v))$$

so this induces a bundle map

$$C : (\mathcal{C}l(X) \otimes \mathbb{C}) \otimes W \to W$$

which we will refer to as the *Clifford multiplication* map yet again.

Finally from Lemma 1.1, $\mathcal{C}l(X)$ contains the subbundle

$$P_{SO(4)} \times_{SO(4)} \mathbb{R}^4 \subset P_{SO(4)} \times_{SO(4)} Cl(\mathbb{R}^4) = Cl(X)$$

which is canonically isomorphic to TX. It follows $Cl(X) \otimes \mathbb{C}$ contains a subbundle canonically isomorphic to $TX \otimes \mathbb{C}$. Thus using the canonical identification of tangent and cotangent bundles, we can define a map

$$C : (T^*X \otimes \mathbb{C}) \otimes W \to W.$$

As a result of Lemma 1.7 , we have the restrictions

$$C : (T^*X \otimes \mathbb{C}) \otimes W^{\pm} \to W^{\mp}.$$

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